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A note on Hauser's type N gravitational field with twist

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Abstract. A sense in which flat space–time can be recovered as a limit of the newly discovered Hauser space–times is discussed. Thus a fairly complete analysis of the symmetries of these space–times, together with the geometry of their linearized principal congruences and Weyl curvature tensors can be made. Unfortunately the Hauser space–times are not asymptotically flat and cannot therefore be regarded as describing gravitational radiation from an isolated system.

1. Introduction

In a recent letter, Hauser (1974) has presented the metric tensor of a two parameter family of exact solutions of Einstein's vacuum gravitational field equation. The Weyl tensor is algebraically special of type $\{4\}$ (or N), so the gravitational field is of the type associated with pure radiation, and the associated congruence of null geodesics has non-vanishing twist. Hauser's solutions are the first explicitly presented solutions with these properties.

In this note we investigate a limiting form of Hauser's metric containing one free parameter, looking at its symmetries, the linearized principal congruence and linearized Weyl tensor. It turns out, unfortunately, that the linearized congruence is not twisting, and both the congruence and the linearized Weyl tensor are singular on certain hypersurfaces in Minkowski's space–time. For one value of the remaining parameter, however, the linearized congruence is the degenerate case of a special type of null congruence which plays a central role in the theory of twistors (Penrose 1967) and is associated with some globally non-singular solutions of the linearized field equation (Penrose 1965, Synge 1956). These latter matters are taken up in the appendix.

From the properties of the linearized fields, we are able to deduce that those Hauser space-times which are not far removed from flat space admit precisely one Killing vector field, and fail to be asymptotically flat.

2. Elementary properties of Hauser's space–time

In order to conform with wider usage (Newman and Penrose 1962), we have altered Hauser's notation, reversed the signature of the metric, and changed the sense of one tetrad vector so that both of the real null tetrad vectors point to the future. The correspondence with Hauser (1974) is given in table 1.

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Table 1.

Hauser	ξ	u	ζ	ρ	σ	z	k	m	t	χ, Δ	τ
Sommers and Walker	ξ	$-r$	ζ	$\sqrt{(2)}X$	$\sqrt{(2)}Y$	ρ	l	$-n$	$-m$	Do not occur	R

With the alterations given in table 1, the metric is given in terms of the null tetrad $\{l^a, n^a, m^a, \bar{m}^a\}$ by $g_{ab} = 2l_{(a}n_{b)} - 2m_{(a}\bar{m}_{b)}$. With respect to a coordinate system $x^a = (\xi, r, \zeta, \bar{\zeta})$, the tetrad has components

$$\begin{aligned} l_a\,dx^a &= R\,d\xi + 2iRX(d\zeta - d\bar{\zeta}) \\ n_a\,dx^a &= dr - 3iR(\bar{A}\,d\zeta - A\,d\bar{\zeta}) \\ m_a\,dx^a &= Al_a\,dx^a + (r - iR)\,d\zeta \end{aligned} \tag{1}$$

where $\zeta =: X + iY$. The functions R and A are expressed in terms of a function f with argument $\xi/(2X^2)$ by

$$\begin{aligned} R &= 2^{3/4}X^{3/2}f \\ A &= X^{-1}[(\xi/(2X^2) - i)f'/f - \tfrac{3}{4}], \end{aligned} \tag{2}$$

where the function f is any solution of the second order, linear, ordinary differential equation

$$16\{1 + [\xi/(2X^2)]^2\}f'' + 3f = 0. \tag{3}$$

The reason that Hauser's parameter Δ does not appear is that it can be set equal to unity by a rescaling of the coordinates together with a boost of the tetrad. Solutions with 'different Δ' are therefore isometric, not just homothetic.

The two parameters labelling the different Hauser solutions are the two parameters in the general solution f of equation (3). It proves convenient to take as basis in the space of solutions of equation (3) the functions $F_{1/4}$ and $F_{3/4}$, where, for large values of the argument $\xi/(2X^2)$, $F_{1/4}$ behaves as $[\xi/(2X^2)]^{1/4}$ and $F_{3/4}$ as $[\xi/(2X^2)]^{3/4}$. In order to facilitate the linearization to be carried out in the next section, we parametrize the solutions with $(\lambda^{1/4}, \mu) \in \mathbf{R}^2$ by writing

$$f = \mu\lambda^{1/4}F_{1/4} + (1-\mu)\lambda^{3/4}F_{3/4}. \tag{4}$$

As was pointed out by Hauser (1974), the metric components in the (ξ, r, X, Y) coordinates do not depend on Y. It follows that the vector field $\partial/\partial Y$ in these coordinates satisfies Killing's equations, ie the associated local one parameter group of local diffeomorphisms consists of isometries. We shall show later that $\partial/\partial Y$ is the only Killing vector field admitted by Hauser's space–times for general values of $(\lambda^{1/4}, \mu)$.

The vector field l^a is tangent to a congruence of non-shearing, affinely parametrized, null geodesics with complex expansion (Newman and Penrose 1962) $\rho = -(r - iR)^{-1}$; the divergence of this congruence is $\frac{1}{2}\nabla_a l^a = r/(r^2 + R^2)$ and its twist is

$$i\bar{m}^a m^b \nabla_{[a}l_{b]} = R/(r^2 + R^2).$$

The scalar r is the affine parameter and the scalars ξ, ζ, $\bar{\zeta}$ are constant along the congruence, ie each choice of values for ξ, ζ, $\bar{\zeta}$ selects a geodesic of the congruence with affine parameter r. The vectorfields n^a, m^a and $\bar{m}^a$ are propagated parallelly along the congruence. The only non-vanishing component of the Weyl curvature tensor in the tetrad $\{l^a, n^a, m^a, \bar{m}^a\}$ is

$$\Psi_4 = C_{abcd} n^a \bar{m}^b n^c \bar{m}^d = 3i\rho[R(\xi + 2iX^2)^2]^{-1}. \tag{5}$$

The quantity Ψ_4 depends on the choice of null tetrad and is not an invariant scalar field on space–time; all algebraic scalar invariants of type $\{4\}$ vacuum curvature tensors vanish identically (cf Penrose 1960). On the other hand, the non-vanishing of Ψ_4 is invariant under tetrad transformations preserving the direction of l^a. This fact will be of importance in our discussion of asymptotic flatness.

The regular part of a Hauser space–time includes all points for which Ψ_4 is finite and the tetrad is defined. Along incomplete and inextendable geodesics, if these occur, either Ψ_4 must become unbounded or the tetrad must cease to be defined, or both. From the expressions (1) and (5) for the tetrad and Ψ_4, we see that this behaviour would require that $R \to 0$, or else both $\xi \to 0$ and $X \to 0$. The latter case also implies $R \to 0$, as may be seen by examining the expression (2) for R, and the asymptotic behaviour (4) of f. Inspecting the r dependence of Ψ_4, one notices the curious feature that Ψ_4 is regular everywhere along a geodesic in the principal null congruence if it is finite for some point of the geodesic. It is therefore of no help to us in analysing singularities that Ψ_4 is a component of the curvature tensor in a tetrad parallelly propagated in the direction of l^a. It seems likely, nevertheless, that geodesics along which $R \to 0$ will be incomplete and inextendable. We shall see later that in the linearized theory, the field is genuinely singular.

3. Linearization of Hauser's metric

We now wish to consider certain limits of this two parameter family of space–times. If we could find the flat Minkowski space–time as a limit of some one parameter subfamily, then we could linearize this subfamily about flat space and learn something about the curved space–times by studying the corresponding linearized field. In fact, as pointed out by Hauser (1974) himself, flat space does occur as a limit of the Hauser family. Because the flat space is recovered in a nontrivial manner from equations (1), we shall discuss the limiting procedure in some detail, following the formulation of Geroch (1969).

Regarding μ as an arbitrary fixed parameter, we consider the family of Hauser space–times corresponding to different values of λ in equation (4). We then have a one parameter family of space–times (M_λ, g_λ) for each μ, and we wish to investigate 'the limit' as $\lambda \to 0$ from above. There is, however, no natural correspondence between the manifolds M_λ for different values of λ. Indeed, as Geroch (1969) has emphasized, inequivalent limits can result from different choices for this correspondence. Hence one cannot speak of the limit as $\lambda \to 0$ of (M_λ, g_λ) *a priori*; one must first make a convention as to how points in the different M_λ are to be identified. Once such a convention has been made, there will exist a diffeomorphism relating M_λ to M_1, say, for each λ. We will then be able to consider the one parameter family of metrics g_λ on the single manifold M_1 and take the limit as $\lambda \to 0$.

What convention should we adopt for identifying points in the different manifolds M_λ of the subfamilies of Hauser space–times we are considering? Making use of the coordinates at hand, one might identify points in the different manifolds if they are labelled by the same values for ξ, r, X, Y. With this convention the metric becomes degenerate as $\lambda \to 0$, so one does not get a regular limit at all!

We choose instead the following identification of points in the different M_λ: the point (ξ, r, X, Y) in M_λ is to be identified for each λ with the point $(\lambda^{1/5}\xi, \lambda^{2/5}r, \lambda^{-2/5}X, \lambda^{-2/5}Y)$ in M_1. The metrics g_λ on M_1 are obtained from the metric (1) for each value of λ in equation (4) by applying the diffeomorphism given explicitly by $\xi \to \lambda^{1/5}\xi$, $r \to \lambda^{2/5}r$, $X \to \lambda^{-2/5}X$, $Y \to \lambda^{-2/5}Y$. With this convention the limiting metric is non-degenerate, due to the fact that the diffeomorphism is compensatingly singular in the limit. In order that the components of the tetrad vectors (1) should also be regular in the limit, we perform a λ-dependent boost on the tetrad:

$$\{l^a, n^a, m^a, \bar{m}^a\} \to \{\lambda^{-2/5}l^a, \lambda^{2/5}n^a, m^a, \bar{m}^a\}.$$

We shall now carry out the limiting procedure explicitly. Recall that

$$f = \mu\lambda^{1/4}F_{1/4} + (1-\mu)\lambda^{3/4}F_{3/4},$$

where the two linearly independent solutions $F_{1/4}$ and $F_{3/4}$ to equation (3) were defined by their asymptotic forms $[\xi/(2X^2)]^{1/4}$ and $[\xi/(2X^2)]^{3/4}$ respectively for large values of their arguments. With the identifications made above, we see that

$$f = \mu\lambda^{1/4}F_{1/4}[\xi/(2\lambda X^2)] + (1-\mu)\lambda^{3/4}F_{3/4}[\xi/(2\lambda X^2)] \tag{6}$$

as a function on M_1, for each λ. If we restrict the allowed range of the coordinates to be such that $\xi/(2X^2)$ remains bounded away from zero, and f has no zeros, then small values of λ imply large values of the argument $\xi/(2X^2)$ of f. It follows that we can insert the asymptotic forms for $F_{1/4}$ and $F_{3/4}$ into f when taking the limit $\lambda \to 0$. The result is

$$R = \lambda^{3/5}\xi^{1/4}[\sqrt{2}\mu X + (1-\mu)\xi^{1/2} + O(\lambda)]$$
$$A = -\tfrac{1}{2}\lambda^{-2/5}[\sqrt{2}\mu + O(\lambda)][\sqrt{2}\mu X + (1-\mu)\xi^{1/2}]^{-1}.$$

It is convenient to note that

$$AR = -\frac{1}{\sqrt{2}}\lambda^{1/5}\xi^{1/4}(\mu + O(\lambda))$$

where the $O(\lambda)$ term does not contribute to the metric to order λ, even when $\mu = 0$. Having these expressions in hand, it is easy to see that the transformed tetrad is given by

$$\begin{aligned} l_a\,dx^a &= [\sqrt{2}\mu X + (1-\mu)\xi^{1/2}]\xi^{1/4}\,d\xi + O(\lambda) \\ n_a\,dx^a &= dr + O(\lambda) \\ m_a\,dx^a &= r\,d\zeta - \frac{1}{\sqrt{2}}\mu\xi^{1/4}\,d\xi + O(\lambda). \end{aligned} \tag{7}$$

The sole component of the curvature tensor in this tetrad is

$$\begin{aligned} \Psi_4 &= C_{abcd}n^a\bar{m}^b n^c\bar{m}^d \\ &= -3i\lambda\{r\xi^{9/4}[\sqrt{2}\mu X + (1-\mu)\xi^{1/2}]\}^{-1} + O(\lambda^2). \end{aligned} \tag{8}$$

Equation (8) demonstrates explicitly that the space–times given by equations (7) are indeed flat when $\lambda = 0$. The flat metric resulting from equations (7) with $\lambda = 0$ has components in the coordinate system (ξ, r, X, Y) given by

$$ds^2 = \xi^{1/4}\, d\xi\{2\sqrt{2}\mu r\, dX + 2[\sqrt{2}\mu X + (1-\mu)\xi^{1/2}]\, dr - \mu^2 \xi^{1/4}\, d\xi\} - 2r^2(dX^2 + dY^2). \qquad (9)$$

4. Analysis of the linearized Hauser metrics

We see from equations (7) that the linearized one-form $l_a\, dx^a$ is proportional to a gradient. Thus l^a is tangent to shear-free null hypersurfaces in flat space–time, the twist having unfortunately gone to zero in the limit $\lambda \to 0$. The primary problem in finding out what this congruence is, as in investigating the singularity structure of the linearized curvature tensor, is to identify the functions ξ, r, X and Y in Minkowski's space–time. As a first step towards a solution of this problem, we note that n^a is a null translational Killing vector in flat space. This can be seen by inverting equations (7) to obtain

$$\begin{aligned} l^a\, \partial/\partial x^a &= \partial/\partial r \\ n^a\, \partial/\partial x^a &= [\sqrt{2}\mu X + (1-\mu)\xi^{1/2}]^{-1}\{\xi^{-1/4}\, \partial/\partial\xi + [\mu/(\sqrt{2}r)]\, \partial/\partial X\} \\ m^a\, \partial/\partial x^a &= -[1/(2r)](\partial/\partial X + i\, \partial/\partial Y) \end{aligned} \qquad (10)$$

and then verifying by direct calculation that $\mathscr{L}_n\, g^{ab} = 0$. Since $n_a\, dx^a = dr$, and n^a is a null Killing vector, we can introduce orthonormal coordinates (t, x, y, z) in Minkowski space to set

$$\sqrt{2}r = t - z. \qquad (11)$$

We next consider the Killing vector field $K^a\, \partial/\partial x^a = \partial/\partial Y$ in coordinates $x^a = (\xi, r, X, Y)$. From equations (9) and (11), the squared norm of K^a is

$$g_{ab}K^aK^b = -2r^2 = -(t-z)^2.$$

Since the components of any Killing vector field in Minkowski's space–time can be simply expressed in terms of an orthonormal coordinate system, it is not difficult to verify that the most general Killing vector field with squared norm $-(t-z)^2$ is

$$a(x\, \partial/\partial t + t\, \partial/\partial x + x\, \partial/\partial z - z\, \partial/\partial x) + b(y\, \partial/\partial t + t\, \partial/\partial y + y\, \partial/\partial z - z\, \partial/\partial y)$$

where $a^2 + b^2 = 1$. Since it is then always possible to rotate the xy axes to achieve $a = 0, b = 1$, we set

$$K^a\, \partial/\partial x^a = \partial/\partial Y = y\, \partial/\partial t + t\, \partial/\partial y + y\, \partial/\partial z - z\, \partial/\partial y \qquad (12)$$

for each value of μ. This choice enables us to identify the function Y since

$$K_a\, dx^a = -2r^2\, dY = -(t-z)^2\, dY$$

from equations (9) and (11), while from equation (12)

$$K_a\, dx^a = y\, d(t-z) - (t-z)\, dy$$

so that

$$dY = d[y/(t-z)].$$

We set

$$Y = y/(t-z). \tag{13}$$

As our third step, we seek an expression for a space-like, unit vector field $\partial/\partial x$ in the coordinates (ξ, r, X, Y), such that $\partial/\partial x$ is orthogonal to and commutes with the vector fields

$$\partial/\partial y = 1/(\sqrt{2}r)\,\partial/\partial Y + \sqrt{2}Yn^a\,\partial/\partial x^a \tag{14}$$

and

$$\partial/\partial t + \partial/\partial z = \sqrt{2}n^a\,\partial/\partial x^a. \tag{15}$$

Expanding $\partial/\partial x$ in the basis $\{l^a, n^a, m^a, \bar{m}^a\}$ and making use of the expressions (7) and (10) for the tetrad results, after some straightforward computation, in

$$\partial/\partial x = \sqrt{2}Xn^a\,\partial/\partial x^a - 1/(\sqrt{2}r)\,\partial/\partial X. \tag{16}$$

Converting the left and right hand sides of equation (16) into one-forms with the metric then allows us to conclude that

$$x = \sqrt{2}rX - \mu\tfrac{4}{5}\xi^{5/4}. \tag{17}$$

Finally, we write the Minkowski metric in the form $\mathrm{d}(t+z)\,\mathrm{d}(t-z) - \mathrm{d}x^2 - \mathrm{d}y^2$ and compare this with equation (9) using equations (11), (13), and (17). The conclusion is that

$$t+z = \sqrt{2}(1-\mu)\tfrac{4}{7}\xi^{7/4} + \sqrt{2}r(X^2+Y^2). \tag{18}$$

What we have found is that for each value of μ, a transformation from Hauser's coordinates (ξ, r, X, Y) for the corresponding linearized solution to an orthonormal coordinate system (t, x, y, z) in Minkowski's space–time is effected by

$$\begin{aligned} &t-z = \sqrt{2}r \qquad t+z = \sqrt{2}(1-\mu)\tfrac{4}{7}\xi^{7/4} + \sqrt{2}r(X^2+Y^2) \\ &x = \sqrt{2}rX - \mu\tfrac{4}{5}\xi^{5/4} \qquad y = \sqrt{2}rY. \end{aligned} \tag{19}$$

We wish now to investigate the linearized principal congruence, that is, the null straight lines in Minkowski's space–time to which the vector field l^a is tangent. We note first of all that $g_{ab}l^aK^b = 0$ so that the Killing vector field K^a is tangent to the null hypersurfaces generated by l^a, K^a being orthogonal to the normal l^a to these hypersurfaces. Now the only shear-free null hypersurfaces in flat space–time are null cones or null hyperplanes, and since the divergence of l^a does not vanish, the hypersurfaces we seek must be null cones. Since the principal congruence must be mapped to itself under the group action induced by K^a, the vertices of these null cones must be fixed points since otherwise the vertices would be slid along the cones by this action, which is absurd. Hence the vertices lie in the plane S: $y = 0, t-z = 0$. Counting degrees of freedom shows that for any given μ these vertices lie on some curve $w(\mu)$ in S. Let us call the curve $w(\mu)$ from which the null cones of the linearized congruence spring, the generating curve for the linearized congruence.

In order to find the generating curve for each μ, we introduce coordinates $v := t+z$, $u := t-z$, x and y. With respect to these coordinates, l^a has components

$$\sqrt{2}(X^2+Y^2, 1, X, Y).$$

But because the integral curves of l^a are straight lines, and flat space has a linear structure in the coordinates (v, u, x, y), l^a must be proportional to the vector having components

$$(v-v_0, u-0, x-x_0, y-0)$$

with proportionality factor $1/r$. Thus the integral curves of l^a spring from the curve

$$w(\mu) = (v_0, 0, x_0, 0)$$

in the plane S. Equating the above expressions for l^a yields

$$v-v_0 = \sqrt{2}r(X^2+Y^2)$$
$$x-x_0 = \sqrt{2}rX$$

and making use of equation (19) then gives the parametric form

$$w(\mu):\mathbf{R} \to M: \xi \to (\sqrt{2}(1-\mu)\tfrac{4}{7}\xi^{7/4}, 0, -\mu\tfrac{4}{5}\xi^{5/4}, 0) \tag{20}$$

for the generating curve in the coordinates (v, u, x, y). The curve $w(\mu)$ is sketched in the x_0v_0 plane for some values of μ in figure 1. Note that the linearized congruence is not defined for points space–like-related to the generating curve $w(\mu)$.

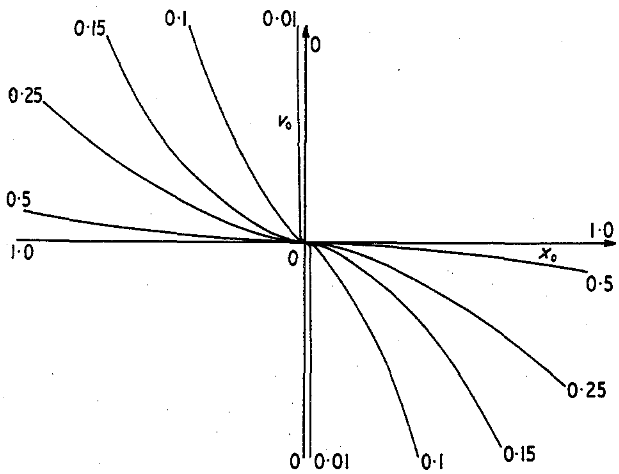

Figure 1. Curves in the x_0v_0 plane from which the linearized principal null congruence springs for some values of μ.

Finally we come to the linearized Weyl tensor or gravitational field. This is given by

$$L_{abcd} = \lim_{\lambda \to 0} \frac{1}{\lambda} C_{abcd}$$

and in the tetrad $\{l^a, n^a, m^a, \overline{m}^a\}$ has the single non-vanishing component

$$\Psi_4^L = L_{abcd}n^a\overline{m}^b n^c\overline{m}^d = -3i\{r\xi^{3/4}[\sqrt{2}\mu X+(1-\mu)\xi^{1/2}]\}^{-1}.$$

In order to locate the singularities of this field, we introduce a constant null tetrad in Minkowski's space–time and examine where the components of L_{abcd} in this tetrad fail to be defined. In our orthonormal coordinate basis the constant tetrad will be $\{\sqrt{2}\,\partial/\partial u, \sqrt{2}\partial/\partial v, \sqrt{2}(\partial/\partial x+\mathrm{i}\,\partial/\partial y), \sqrt{2}(\partial/\partial x-\mathrm{i}\,\partial/\partial y)\}$. The linearized Hauser tetrad is obtained by a null rotation of this constant tetrad by an amount $\zeta = X+\mathrm{i}Y$ about the direction $\sqrt{2}\,\partial/\partial v$. This may be verified using equations (10) and (19). It then follows that the five complex components of L_{abcd} in the constant basis are $\Psi_4^L\zeta^\alpha$, $\alpha = 0, 1, 2, 3, 4$.

To locate the regions of flat space where the components are not defined, we re-express Ψ_4^L as

$$\Psi_4^L = -3\mathrm{i}[r\xi^2(l^a w_a)]^{-1}.$$

Here w^a is the tangent to the generating curve (20) parametrized by ξ. The product $l^a w_a$ vanishes on the boundary of the causal future and past of the generating curve w. This includes the hyperplane $r = 0$. The hypersurface $\xi = 0$ is half of the null cone of the origin $v = u = x = y = 0$ for $\mu \neq 0$. (If $\mu = 0$, the hypersurface $\xi = 0$ is the entire null cone of the origin.) The function ζ is singular on the hyperplane $t-z = 0$ (ie $r = 0$) but is regular at all other finite points. We conclude, therefore, that the linearized field is singular on the boundary of the future and past of the generating curve and on that portion of the origin's null cone which is generated by the principal null congruence.

We note that the linearized Hauser fields do not peel off along outgoing null directions in the manner appropriate for linearized asymptotically flat fields. For example, going out along the direction $\sqrt{2}\,\partial/\partial v$, the function ζ grows without bound so that components of the curvature in the constant basis also grow without bound in that null direction.

5. Conclusion

We have been able to make a fairly complete analysis of the linearized Hauser space-times. On the basis of this analysis, we wish to draw some conclusions concerning the nature of the non-flat, exact Hauser solutions. It is, of course, not true that all features of the linearized approximations will carry over to the exact solutions. Nevertheless, one might expect that certain qualitative properties of a linearized field will continue to hold in some open neighbourhood of flat space along the curve of exact solutions to which the linearized solution is tangent. The properties we propose to establish for the exact Hauser solutions, in some neighbourhood of flat space–time, are the uniqueness of the Killing vector field K^a, and the failure of asymptotic flatness.

Suppose there are two Killing vectors in the exact Hauser space–times, the one found by Hauser that we have called K^a and a second one called J^a. Since J^a must preserve the principal congruence, let us scale l^a so that the commutator $[J, l]^a = 0$. Now consider a one parameter family of these Hauser space–times with flat space as a limit. From Geroch's (1969) work on limits of space–times, we know that J^a is a Minkowski Killing vector field in the limit and is distinct from the limit

$$y\,\partial/\partial t+t\,\partial/\partial y+y\,\partial/\partial z-z\,\partial/\partial y$$

of K^a. Since $[J, l]^a = 0$ for all the space–times, it must be true by continuity also in the limit. So we must look for another Minkowski Killing vector field preserving the linearized congruence. Such a vector field must leave invariant the curve from which sprang the null cones of the linearized congruence. Except for $\mu = 0$ or 1, this curve is only preserved if its points are fixed points of the group action, and it is clear that the Killing vector field must then have fixed points everywhere on the plane $y = 0 = t-z$

But the only such Killing vector field is the one we already know about. We conclude that $J^a = K^a$.

We shall now show that asymptotic flatness of the Hauser space–times is inconsistent with the combination of their algebraic type and the action of their isometry group. The argument is an adaptation of an idea of Penrose's (private communication). We examine a one parameter family of Hauser space–times with a Minkowski limit. Supposing the Hauser space–times to be asymptotically flat, we would obtain a family of null infinities $\mathscr{I}$ (see eg Penrose 1974) with Minkowski's $\mathscr{I}$ in the limit. Since the unique Killing vector field K^a does not induce closed orbits on Minkowski's $\mathscr{I}$, it must induce open orbits on the $\mathscr{I}$ of the Hauser space–times in some open neighbourhood of flat space. That identifications which would close the orbits of K^a on $\mathscr{I}$ in Minkowski's space–time cannot be made without violating either causality or regularity of the manifold is obvious from the fact that K^a is the sum of a boost and a rotation; such identifications will also not be possible in some neighbourhood of flat space. It follows that the induced action of K^a on $\mathscr{I}$ will map some section of $\mathscr{I}$ onto a future section. Since K^a is an isometry of space–time, the mass integral (see eg Penrose 1974) must give the same value on both sections. Now the change in mass due to gravitational radiation can be expressed as an integral of the squared modulus of the news function over the volume of $\mathscr{I}$ bounded by the two sections. But since the change in mass is zero, the news function must vanish everywhere between the two sections. In particular, the radiation field on $\mathscr{I}$, which is a derivative of the news function, must vanish between the sections. In order to establish a contradiction with the assumption of asymptotic flatness, it remains to show that the radiation field for a non-flat type $\{4\}$ space–time does not vanish between two sections of $\mathscr{I}$ which are mapped into one another under K^a.

Consider some integral curve of the repeated principal null vector l^a. This is a null geodesic, and hence meets $\mathscr{I}$ in some point p. Except, perhaps, for isolated exceptional points, it will then always be possible to choose a pair of non-intersecting sections of $\mathscr{I}$ which sandwich the point p, and which are mapped into each other under the action of the group. Since a finite null rotation about l^a will align n^a with the null direction in $\mathscr{I}$ at p, and since any conformal factor Ω defining $\mathscr{I}$ has the behaviour $1/r$ near $\mathscr{I}$, r being an affine parameter along the chosen null geodesic, it follows that the radiation field will not vanish at p. By continuity, the radiation field will not then vanish in some open neighbourhood of p, which establishes the required contradiction.

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Appendix

Among the one parameter family of linearized Hauser solutions, the one for which $\mu = 0$ deserves special attention. From equation (20) it is seen that the generating curves for the congruences all have space-like tangents except in this case; the linearized congruence here consists of the null lines of Minkowski space–time which meet the null line $x = y = u = 0$. We confer special attention on this case because linearized null fields with this congruence, as we shall demonstrate, also arise as limits of globally regular linearized null fields which peel off at $\mathscr{I}$ in the way appropriate for asymptotically flat gravitational fields.

We first review a technique (Synge 1956, Penrose 1965) for constructing a globally non-singular null solution of the linearized gravitational field equation, which peels off appropriately in all directions in flat space–time. The solution is constructed from a constant spinor field ι^A together with a globally regular complex function χ satisfying the two equations

$$\nabla^a \nabla_a \chi = 0$$

and (A.1)

$$\nabla_a \nabla_b \chi^{-1} = \eta_{ab}$$

where the ∇_a is the flat affine connection and η_{ab} the Minkowski metric. The linearized gravitational field is

$$\Psi_{ABCD} = \bar{\iota}^{A'} \bar{\iota}^{B'} \bar{\iota}^{C'} \bar{\iota}^{D'} \nabla_{AA'} \nabla_{BB'} \nabla_{CC'} \nabla_{DD'} \chi.$$

Exploiting the constancy of $\bar{\iota}^{A'}$, the commutativity of derivatives, and the wave equation satisfied by χ, the satisfaction of the field equation is apparent:

$$\nabla^{DE'} \Psi_{ABCD} = 0.$$

In a similar way one verifies that

$$\Psi_{ABCD} \bar{\iota}^{E'} \nabla^{D}_{E'} \chi^{-1} = 0$$

from which it follows that

$$\kappa_A := \bar{\iota}^{A'} \nabla_{AA'} \chi^{-1} \qquad \text{(A.2)}$$

is the fourfold principal spinor of Ψ_{ABCD}.

The functions χ^{-1} satisfying equations (A.1) are the constant multiples of

$$\chi^{-1} = \tfrac{1}{2}(x^a + i\theta^a)(x_a + i\theta_a). \qquad \text{(A.3)}$$

Here x^a is the Minkowski position vector and θ^a is a constant vector field which is time-like in order that χ be everywhere regular.

Taking advantage of the explicit expression for χ^{-1}, we can now characterize the principal null direction $\kappa^A \bar{\kappa}^{A'}$. From equations (A.2) and (A.3) it is immediate that

$$\nabla_{A'A} \kappa_B = -\bar{\iota}_{A'} \epsilon_{AB}$$

where ϵ_{AB} is the antisymmetric spinor satisfying $\eta_{ab} = \epsilon_{AB} \epsilon_{A'B'}$. The spinor field κ_A is therefore a solution of the twistor equation (Penrose 1967)

$$\nabla^{(A}_{A'} \kappa^{B)} = 0$$

and the pair $(\kappa^A, \bar{\iota}_{A'})$ represent a twistor Z^α. The congruence of null straight lines to which $\kappa^A \bar{\kappa}^{A'}$ is tangent is called a Robinson congruence (Penrose 1967). The null fields described here are examples of 'elementary states' (Penrose 1968) in twistor theory.

A null linearized gravitational field may also be constructed in the above manner with θ^a no longer time-like but coincident with $n^a = \iota^A \bar{\iota}^{A'}$ instead. By equation (A.2) the principal spinor is then given by

$$\kappa_A = \bar{\iota}^{A'} x_{AA'}.$$

from which it follows that the principal null vector field k^a has the form

$$k^a = (\theta^b x_b)x^a - \tfrac{1}{2}(x^b x_b)\theta^a.$$

This vector addition equation shows that the principal null vectors $(\theta^b x_b)^{-1}k^a$ join the space-time points x^a to points on the null line through the origin in the direction of θ^a. The congruence is therefore the same as the linearized Hauser congruence for $\mu = 0$.

Since the principal congruences agree, one may wonder if the linearized Hauser field agrees with the degenerate elementary state field described here. Given a field $\Psi_{ABCD} = \psi\kappa_A\kappa_B\kappa_C\kappa_D$ satisfying $\nabla^{AA'}\Psi_{ABCD} = 0$, it is evident that $\alpha\Psi_{ABCD}$ is also a solution whenever α satisfies

$$\kappa^A\nabla_{AA'}\alpha = 0.$$

When $k^a = \kappa^A\bar{\kappa}^{A'}$ is hypersurface orthogonal, α may, in particular, be any function which is constant on the null hypersurfaces. The linearized Hauser field, in fact, differs from the degenerate elementary state field by just such a factor.

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